

INVARIANCE MEASURES OF STOCHASTIC 2D NAVIER-STOKES EQUATIONS DRIVEN BY α -STABLE PROCESSES

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ABSTRACT. In this note we prove the well-posedness for stochastic 2D Navier-Stokes equation driven by general Lévy processes (in particular, α -stable processes), and obtain the existence of invariant measures.

1. INTRODUCTION AND MAIN RESULT

In this article we are concerned with the following stochastic 2D Navier-Stokes equation in torus $\mathbb{T}^2 = (0, 1]^2$:

$$du_t = [\Delta u_t - (u_t \cdot \nabla)u_t + \nabla p_t]dt + dL_t, \quad \operatorname{div} u_t = 0, \quad u_0 = \varphi \in \mathbb{H}^0, \quad (1.1)$$

where $u_t(x) = (u_t^1(x), u_t^2(x))$ is the 2D-velocity field, p is the pressure, and $(L_t)_{t \geq 0}$ is an infinite dimensional cylindrical Lévy process given by

$$L_t = \sum_{j \in \mathbb{N}} \beta_j L_t^{(j)} e_j,$$

where $\{(L_t^{(j)})_{t \geq 0}, j \in \mathbb{N}\}$ is a sequence of independent one dimensional purely discontinuous Lévy processes defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}; P)$ and with the same Lévy measure ν , $\{\beta_j, j \in \mathbb{N}\}$ is a sequence of positive numbers and $\{e_j, j \in \mathbb{N}\}$ is a sequence of orthogonal basis of Hilbert space \mathbb{H}^0 , where for $\gamma \in \mathbb{R}$, \mathbb{H}^γ with the norm $\|\cdot\|_\gamma$ and inner product $\langle \cdot, \cdot \rangle_\gamma$ denotes the usual Sobolev space of divergence free vector fields on \mathbb{T}^2 (see Section 2 for a definition).

As a continuous model, stochastic Navier-Stokes equation driven by Brownian motion has been extensively studied in the past decades (cf. [9, 3, 5, 8], etc.). Meanwhile, stochastic partial differential equation with jump has also been studied recently (cf. [12, 6]). However, in the well-known results, the assumption that the jump process has finite second order moments was required in order to obtain the usual energy estimate. This excludes the interest α -stable process. In this note, we establish the well posedness for stochastic 2D Navier-Stokes equation (1.1) driven by a general cylindrical Lévy process, and obtain the existence of invariant measures for this discontinuous model. More precisely, we shall prove that:

Theorem 1.1. *Suppose that for some $\theta \in (0, 1]$,*

$$(\mathbf{H}_\theta): \quad H_\theta := \int_{|x|>1} |x|^\theta \nu(dx) + \sum_{j \in \mathbb{N}} |\beta_j|^\theta < +\infty.$$

Then for any $\varphi \in \mathbb{H}^0$, there exists a unique solution $(u_t)_{t \geq 0} = (u_t(\varphi))_{t \geq 0}$ to equation (1.1) satisfying that for almost all ω and for any $t > 0$,

- (i) $t \mapsto u_t(\omega)$ is right continuous and has left-hand limit in \mathbb{H}^0 , and $\int_0^t \|\nabla u_s(\omega)\|_0^2 ds < +\infty$;
- (ii) it holds that for any $\phi \in \mathbb{H}^1$,

$$\langle u_t(\omega), \phi \rangle_0 = \langle \varphi, \phi \rangle_0 + \int_0^t [\langle \Delta u_s(\omega), \phi \rangle_0 + \langle u_s(\omega) \otimes u_s(\omega), \nabla \phi \rangle_0] ds + \langle L_t(\omega), \phi \rangle_0.$$

Moreover, there exists a constant $C = C(H_\theta, \theta) > 0$ such that for any $t > 0$,

$$\mathbb{E} \left(\sup_{s \in [0, t]} \|u_s\|_0^\theta \right) + \mathbb{E} \left(\int_0^t \frac{\|\nabla u_s\|_0^2}{(\|u_s\|_0^2 + 1)^{1-\theta/2}} ds \right) \leq C(1 + \|\varphi\|_0^\theta + t). \quad (1.2)$$

In particular, there exists a probability measure μ on $(\mathbb{H}^0, \mathcal{B}(\mathbb{H}^0))$ called invariant measure of $(u_t(\varphi))_{t \geq 0}$ such that for any bounded measurable function Φ on \mathbb{H}^0 ,

$$\int_{\mathbb{H}^0} \mathbb{E} \Phi(u_t(\varphi)) \mu(d\varphi) = \int_{\mathbb{H}^0} \Phi(\varphi) \mu(d\varphi).$$

Remark 1.2. Assumption (\mathbf{H}_θ) implies that cylindrical Lévy process $(L_t)_{t \geq 0}$ admits a cadlag version in \mathbb{H}^0 and for any $t > 0$ (cf. [13, p.159, Theorem 25.3]),

$$\mathbb{E} \|L_t\|_0^\theta < +\infty.$$

In fact, for $\theta \in (0, 1]$, by the elementary inequality $(a + b)^\theta \leq a^\theta + b^\theta$, we have

$$\mathbb{E} \|L_t\|_0^\theta \leq \mathbb{E} \left(\sum_{j \in \mathbb{N}} |\beta_j| \cdot |L_t^{(j)}| \right)^\theta \leq \sum_{j \in \mathbb{N}} |\beta_j|^\theta \cdot \mathbb{E} |L_t^{(j)}|^\theta = \mathbb{E} |L_t^{(1)}|^\theta \sum_{j \in \mathbb{N}} |\beta_j|^\theta < +\infty.$$

Remark 1.3. By estimate (1.2) and Poincaré's inequality, we have

$$\begin{aligned} \mathbb{E} \left(\int_0^t \|\nabla u_s\|_0^\theta ds \right) &\leq \mathbb{E} \left(\int_0^t \frac{\|\nabla u_s\|_0^\theta (\|u_s\|_0^{2-\theta} + 1)}{(\|u_s\|_0^2 + 1)^{1-\theta/2}} ds \right) \\ &\leq C \mathbb{E} \left(\int_0^t \frac{\|\nabla u_s\|_0^2 + 1}{(\|u_s\|_0^2 + 1)^{1-\theta/2}} ds \right) \\ &\leq C(1 + \|\varphi\|_0^\theta + t). \end{aligned}$$

This estimate in particular yields the existence of invariant measures by the classical Bogoliubov-Krylov's argument (cf. [4]).

Remark 1.4. An obvious open question is about the uniqueness of invariant measures (i.e. ergodicity) for discontinuous system (1.1). The notion of asymptotic strong Feller property in [9] is perhaps helpful for solving this problem.

This paper is organized as follows: In Section 2, we give some necessary materials. In Section 3, we prove the main result.

2. PRELIMINARIES

In this section we prepare some materials for later use. Let $C_0^\infty(\mathbb{T}^2)^2$ be the space of all smooth \mathbb{R}^2 -valued function on \mathbb{T}^2 with vanishing mean and divergence, i.e.,

$$\int_{\mathbb{T}^2} f(x) dx = 0, \quad \operatorname{div} f(x) = 0.$$

For $\gamma \in \mathbb{R}$, let \mathbb{H}^γ be the completion of $C_0^\infty(\mathbb{T}^2)^2$ with respect to the norm

$$\|f\|_\gamma = \left(\int_{\mathbb{T}^2} |(-\Delta)^{\gamma/2} f(x)|^2 dx \right)^{1/2},$$

where $(-\Delta)^{\gamma/2}$ is defined through Fourier's transform. Thus, $(\mathbb{H}^\gamma, \|\cdot\|_\gamma)$ is a separable Hilbert space with the obvious inner product

$$\langle f, g \rangle_\gamma := \int_{\mathbb{T}^2} (-\Delta)^{\gamma/2} f(x) \cdot (-\Delta)^{\gamma/2} g(x) dx.$$

Below, we shall fix an orthogonal basis $\{e_j, j \in \mathbb{N}\} \subset C_0^\infty(\mathbb{T}^2)^2$ of \mathbb{H}^0 consisting of the eigenvectors of Δ , i.e.,

$$\Delta e_j = -\lambda_j e_j, \quad \langle e_j, e_j \rangle_0 = 1, \quad j = 1, 2, \dots, \quad (2.1)$$

where $0 < \lambda_1 < \dots < \lambda_j \uparrow \infty$.

Let $\{(L_t^{(j)})_{t \geq 0}, j \in \mathbb{N}\}$ be a sequence of independent one dimensional purely discontinuous Lévy processes with the same characteristic function, i.e.,

$$\mathbb{E} e^{i\xi L_t^{(j)}} = e^{-t\psi(\xi)}, \quad \forall t \geq 0, j = 1, 2, \dots,$$

where $\psi(\xi)$ is a complex valued function called Lévy symbol given by

$$\psi(\xi) = \int_{\mathbb{R} \setminus \{0\}} (e^{i\xi y} - 1 - i\xi y 1_{|y| \leq 1}) \nu(dy),$$

where ν is the Lévy measure and satisfies that

$$\int_{\mathbb{R} \setminus \{0\}} 1 \wedge |y|^2 \nu(dy) < +\infty.$$

For $t > 0$ and $\Gamma \in \mathcal{B}(\mathbb{R} \setminus \{0\})$, the Poisson random measure associated with $L_t^{(j)}$ is defined by

$$N^{(j)}(t, \Gamma) := \sum_{s \in (0, t]} 1_\Gamma(L_s^{(j)} - L_{s-}^{(j)}).$$

The compensated Poisson random measure is given by

$$\tilde{N}^{(j)}(t, \Gamma) = N^{(j)}(t, \Gamma) - t\nu(\Gamma).$$

By Lévy-Itô's decomposition (cf. [2, p.108, Theorem 2.4.16]), one has

$$L_t^{(j)} = \int_{|x| \leq 1} x \tilde{N}^{(j)}(t, dx) + \int_{|x| > 1} x N^{(j)}(t, dx).$$

For a Polish space (\mathbb{G}, ρ) , let $\mathbb{D}(\mathbb{R}_+; \mathbb{G})$ be the space of all right continuous functions with left-hand limits from \mathbb{R}_+ to \mathbb{G} , which is endowed with the Skorohod topology:

$$d_{\mathbb{G}}(u, v) := \inf_{\lambda \in \Lambda} \left[\sup_{s \neq t} \left| \log \frac{\lambda(t) - \lambda(s)}{t - s} \right| \vee \int_0^\infty \sup_{t \geq 0} (\rho(u_{t \wedge r}, v_{\lambda(t) \wedge r}) \wedge 1) e^{-r} dr \right], \quad (2.2)$$

where Λ is the space of all continuous and strictly increasing function from $\mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $\lambda(0) = 0$ and $\lambda(\infty) = \infty$. Thus, $(\mathbb{D}(\mathbb{R}_+; \mathbb{G}); d)$ is again a Polish space (cf. [7, p.121, Theorem 5.6]).

We need the following tightness criterion, which is a direct combination of [11, Corollary 5.2] and Aldous's criterion [1].

Theorem 2.1. *Let $\{(X_t^n)_{t \geq 0}, n \in \mathbb{N}\}$ be a sequence of \mathbb{H}^{-1} -valued stochastic processes with cadlag path. Assume that*

- (i) *for each $\phi \in C_0^\infty(\mathbb{T}^2)^2$ and $t > 0$, $\lim_{K \rightarrow \infty} \sup_{n \in \mathbb{N}} P\left\{ \sup_{s \in [0, t]} |\langle X_s^n, \phi \rangle_{-1}| \geq K \right\} = 0$;*
- (ii) *for each $\phi \in C_0^\infty(\mathbb{T}^2)^2$ and $t, a > 0$, $\lim_{\varepsilon \rightarrow 0+} \sup_{n \in \mathbb{N}} \sup_{\tau \in \mathcal{S}_t} P\left\{ |\langle X_\tau^n - X_{\tau+\varepsilon}^n, \phi \rangle_{-1}| \geq a \right\} = 0$,
where \mathcal{S}_t denotes all the (\mathcal{F}_t) -stopping times with bound t ;*
- (iii) *for every $\varepsilon > 0$ and $t > 0$,*

$$\lim_{m \rightarrow \infty} \sup_{n \in \mathbb{N}} P\left(\sup_{s \in [0, t]} \sum_{j=m}^\infty \langle X_s^n, e_j \rangle_{-1}^2 \geq \varepsilon \right) = 0.$$

Then the laws of $(X_t^n)_{t \geq 0}$ in $\mathbb{D}(\mathbb{R}_+; \mathbb{H}^{-1})$ is tight.

The following result comes from [7, p.131 Theorem 7.8].

Theorem 2.2. Suppose that stochastic processes sequence $\{(X_t^n)_{t \geq 0}, n \in \mathbb{N}\}$ weakly converges to $(X_t)_{t \geq 0}$ in $\mathbb{D}(\mathbb{R}_+; \mathbb{H}^{-1})$. Then, for any $t > 0$ and $\phi \in \mathbb{H}^1$, there exists a sequence $t_n \downarrow t$ such that for any bounded continuous function f ,

$$\lim_{n \rightarrow \infty} \mathbb{E}f(\langle X_{t_n}^n, \phi \rangle_{-1}) = \mathbb{E}f(\langle X_t, \phi \rangle_{-1}).$$

We also need the following technical result.

Lemma 2.3. Suppose that sequence $\{u^n, n \in \mathbb{N}\}$ converges to u in $\mathbb{D}(\mathbb{R}_+; \mathbb{H}^{-1})$. Then for any $T > 0$ and $m \in \mathbb{N}$,

$$\sup_{t \in [0, T]} \|u_t\|_0 \leq \liminf_{n \rightarrow \infty} \sup_{t \in [0, T + \frac{1}{m}]} \|u_t^n\|_0. \quad (2.3)$$

If in addition, for Lebesgue almost all t , u_t^n converges to u_t in \mathbb{H}^0 , then for any $\beta > 0$,

$$\int_0^T \frac{\|\nabla u_t\|_0^2}{(1 + \|u_t\|_0^2)^\beta} dt \leq \liminf_{n \rightarrow \infty} \int_0^T \frac{\|\nabla u_t^n\|_0^2}{(1 + \|u_t^n\|_0^2)^\beta} dt. \quad (2.4)$$

Proof. Without loss of generality, we assume that the right hand side of (2.3) is finite. For any $\phi \in \mathbb{H}^1$, it is clear that $t \mapsto \langle u_t, \phi \rangle_0$ is a cadlag real valued function, and by definition (2.2) of Skorohod metric, we have

$$d_{\mathbb{R}}(\langle u^n, \phi \rangle_0, \langle u, \phi \rangle_0) \leq (2 + \|\phi\|_1) d_{\mathbb{H}^{-1}}(u^n, u),$$

and so $\langle u^n, \phi \rangle_0$ converges to $\langle u, \phi \rangle_0$ in $\mathbb{D}(\mathbb{R}_+; \mathbb{R})$ as $n \rightarrow \infty$. Since the discontinuous points of $\langle u, \phi \rangle_0$ are at most countable, for any $T > 0$ and $m \in \mathbb{N}$, there exists a time $T_m \in (T, T + 1/m)$ such that $\langle u, \phi \rangle_0$ is continuous at T_m . Thus, we have (cf. [7, p.119, Proposition 5.3])

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T_m]} |\langle u_t^n, \phi \rangle_0| = \sup_{t \in [0, T_m]} |\langle u_t, \phi \rangle_0|.$$

Hence,

$$\begin{aligned} \sup_{t \in [0, T]} \|u_t\|_0 &= \sup_{t \in [0, T]} \sup_{\phi \in \mathbb{H}^1; \|\phi\|_0 \leq 1} |\langle u_t, \phi \rangle_0| \\ &\leq \sup_{\phi \in \mathbb{H}^1; \|\phi\|_0 \leq 1} \sup_{t \in [0, T_m]} |\langle u_t, \phi \rangle_0| \\ &= \sup_{\phi \in \mathbb{H}^1; \|\phi\|_0 \leq 1} \lim_{n \rightarrow \infty} \sup_{t \in [0, T_m]} |\langle u_t^n, \phi \rangle_0| \\ &\leq \liminf_{n \rightarrow \infty} \sup_{\phi \in \mathbb{H}^1; \|\phi\|_0 \leq 1} \sup_{t \in [0, T_m]} |\langle u_t^n, \phi \rangle_0| \\ &= \liminf_{n \rightarrow \infty} \sup_{t \in [0, T_m]} \|u_t^n\|_0. \end{aligned}$$

Thus, (2.3) is proven.

For proving (2.4), let \mathcal{N} be the Lebesgue null set such that for all $t \notin \mathcal{N}$, u_t^n converges to u_t in \mathbb{H}^0 . Fixing a $t \notin \mathcal{N}$, then as above, we have

$$\frac{\|\nabla u_t\|_0^2}{(1 + \|u_t\|_0^2)^\beta} \leq \frac{\liminf_{n \rightarrow \infty} \|\nabla u_t^n\|_0^2}{(1 + \lim_{n \rightarrow \infty} \|u_t^n\|_0^2)^\beta} \leq \liminf_{n \rightarrow \infty} \frac{\|\nabla u_t^n\|_0^2}{(1 + \|u_t^n\|_0^2)^\beta}.$$

Estimate (2.4) now follows by Fatou's lemma. \square

3. PROOF OF THEOREM 1.1

We first give the following definition about the weak solutions to equation (1.1).

Definition 3.1. A probability measure P on $\mathbb{D}(\mathbb{R}_+; \mathbb{H}^{-1})$ is called a weak solution of equation (1.1) if

- (i) for any $t > 0$, $P(u \in \mathbb{D}(\mathbb{R}_+; \mathbb{H}^{-1}) : \sup_{s \in [0, t]} \|u_s\|_0 + \int_0^t \|\nabla u_s\|_0^2 ds < +\infty) = 1$;
- (ii) for any $j \in \mathbb{N}$,

$$M_t^{(j)}(u) := \langle u_t, e_j \rangle_0 - \langle u_0, e_j \rangle_0 - \int_0^t [\langle u_s, \Delta e_j \rangle_0 + \langle u_s \otimes u_s, \nabla e_j \rangle_0] ds \quad (3.1)$$

is a Lévy process with the characteristic function

$$\mathbb{E} e^{i\xi M_t^{(j)}} = \exp \left\{ t \int_{\mathbb{R} \setminus \{0\}} (e^{i\xi y \beta_j} - 1 - i\xi y \beta_j 1_{|y| \leq 1}) \nu(dy) \right\},$$

and $\{(M_t^{(j)})_{t \geq 0}, j \in \mathbb{N}\}$ is a sequence of independent Lévy processes.

Proof of Existence of Weak Solutions: We use Galerkin's approximation to prove the existence of weak solutions and divide the proof into three steps.

(Step 1): For $n \in \mathbb{N}$, set

$$\mathbb{H}_n^0 := \text{span}\{e_1, e_2, \dots, e_n\},$$

and let Π_n be the projection from \mathbb{H}^0 to \mathbb{H}_n^0 and define

$$L_t^n := \sum_{j=1}^n \beta_j L_t^{(j)} e_j = \sum_{j=1}^n \int_{|y| \leq 1} y \beta_j e_j \tilde{N}^{(j)}(t, dy) + \sum_{j=1}^n \int_{|y| > 1} y \beta_j e_j N^{(j)}(t, dy).$$

Consider the following finite dimensional SDE driven by finite dimensional Lévy process L_t^n :

$$du_t^n = [\Delta u_t^n - \Pi_n((u_t^n \cdot \nabla) u_t^n)] dt + dL_t^n, \quad u_0^n = \Pi_n \varphi. \quad (3.2)$$

Since for any $R > 0$ and $u, v \in \mathbb{H}_n^0$ with $\|u\|_0, \|v\|_0 \leq R$,

$$\|\Pi_n((u \cdot \nabla) u - (v \cdot \nabla) v)\|_0 \leq C_{R,n} \|u - v\|_0$$

and

$$\langle u, \Delta u - \Pi_n((u \cdot \nabla) u) \rangle_0 = -\|\nabla u\|_0, \quad \forall u \in \mathbb{H}_n^0, \quad (3.3)$$

finite dimensional SDE (3.2) is thus well-posed.

Define a smooth function f_n on \mathbb{H}_n^0 by

$$f_n(u) := (\|u\|_0^2 + 1)^{\theta/2}, \quad u \in \mathbb{H}_n^0.$$

By simple calculations, we have

$$\nabla f_n(u) = \frac{\theta u}{(\|u\|_0^2 + 1)^{1-\theta/2}}, \quad \nabla^2 f_n(u) = \frac{\theta \sum_{i=1}^n e_i \otimes e_i}{(\|u\|_0^2 + 1)^{1-\theta/2}} - \frac{\theta(2-\theta)u \otimes u}{(\|u\|_0^2 + 1)^{2-\theta/2}}, \quad (3.4)$$

and for all $u, v \in \mathbb{H}_n^0$,

$$|f_n(u) - f_n(v)| \leq |(\|u\|_0^2 + 1)^{1/2} - (\|v\|_0^2 + 1)^{1/2}|^\theta \leq \|u - v\|_0^\theta. \quad (3.5)$$

By (3.2), (3.3), (3.4) and Itô's formula (cf. [2, p.226, Theorem 4.4.7]), we have

$$\begin{aligned} f_n(u_t^n) &= f_n(u_0^n) - \int_0^t \frac{\theta \|\nabla u_s^n\|_0^2}{(\|u_s^n\|_0^2 + 1)^{1-\theta/2}} ds + \sum_{j=1}^n \int_0^t \int_{|y| \leq 1} [f_n(u_s^n + y \beta_j e_j) - f_n(u_s^n)] \tilde{N}^{(j)}(ds, dy) \\ &\quad + \sum_{j=1}^n \int_0^t \int_{|y| \leq 1} \left[f_n(u_s^n + y \beta_j e_j) - f_n(u_s^n) - \frac{\theta \langle u_s^n, y \beta_j e_j \rangle_0}{(\|u_s^n\|_0^2 + 1)^{1-\theta/2}} \right] \nu(dy) ds \end{aligned}$$

$$\begin{aligned}
& + \sum_{j=1}^n \int_0^t \int_{|y|>1} [f_n(u_s^n + y\beta_j e_j) - f_n(u_s^n)] N^{(j)}(ds, dy) \\
& =: f_n(u_0^n) - I_1^n(t) + I_2^n(t) + I_3^n(t) + I_4^n(t).
\end{aligned}$$

For $I_2^n(t)$, by Burkholder's inequality and (3.5), we have

$$\begin{aligned}
\mathbb{E} \left(\sup_{t \in [0, T]} I_2^n(t) \right) & \leq C \sum_{j=1}^n \mathbb{E} \left(\int_0^T \int_{|y| \leq 1} |f_n(u_s^n + y\beta_j e_j) - f_n(u_s^n)|^2 N^{(j)}(ds, dy) \right)^{1/2} \\
& \leq C \sum_{j=1}^n \left(\mathbb{E} \int_0^T \int_{|y| \leq 1} |f_n(u_s^n + y\beta_j e_j) - f_n(u_s^n)|^2 \nu(dy) ds \right)^{1/2} \\
& \leq CT^{1/2} \sum_{j=1}^n |\beta_j|^\theta \left(\int_{|y| \leq 1} |y|^{2\theta} \nu(dy) \right)^{1/2} \\
& \leq CT^{1/2} \sum_{j=1}^\infty |\beta_j|^\theta \left(\int_{|y| \leq 1} |y|^2 \nu(dy) \right)^{1/2}.
\end{aligned}$$

where we have used condition (\mathbf{H}_θ) . Here and after, the constant C is independent of n and T . For $I_3^n(t)$, by Taylor's expansion and (3.4), we have

$$\mathbb{E} \left(\sup_{t \in [0, T]} I_3^n(t) \right) \leq C \sum_{j=1}^n \beta_j^2 \int_0^T \int_{|y| \leq 1} |y|^2 \nu(dy) ds \leq CT \sum_{j=1}^\infty |\beta_j|^\theta \int_{|y| \leq 1} |y|^2 \nu(dy).$$

For $I_4^n(t)$, by (3.5), we have

$$\begin{aligned}
\mathbb{E} \left(\sup_{t \in [0, T]} I_4^n(t) \right) & \leq \sum_{j=1}^n \mathbb{E} \left(\int_0^T \int_{|y|>1} |f_n(u_s^n + y\beta_j e_j) - f_n(u_s^n)| N^{(j)}(ds, dy) \right) \\
& = \sum_{j=1}^n \mathbb{E} \left(\int_0^T \int_{|y|>1} |f_n(u_s^n + y\beta_j e_j) - f_n(u_s^n)| \nu(dy) ds \right) \\
& \leq CT \sum_{j=1}^\infty |\beta_j|^\theta \int_{|y|>1} |y|^\theta \nu(dy).
\end{aligned}$$

Combining the above calculations, we obtain that

$$\mathbb{E} \left(\sup_{t \in [0, T]} (\|u_t^n\|_0^2 + 1)^{\theta/2} \right) + \mathbb{E} \int_0^T \frac{\theta \|\nabla u_s^n\|_0^2}{(\|u_s^n\|_0^2 + 1)^{1-\theta/2}} ds \leq (\|\varphi\|_0^2 + 1)^{\theta/2} + CT + CT^{1/2}. \quad (3.6)$$

(Step 2): In this step, we use Theorem 2.1 to show that $\{(u_t^n)_{t \geq 0}, n \in \mathbb{N}\}$ is tight in $\mathbb{D}(\mathbb{R}_+; \mathbb{H}^{-1})$. For any $\phi \in C_0^\infty(\mathbb{T}^2)^2$, by equation (3.2), we have

$$\begin{aligned}
\langle u_t^n, \phi \rangle_{-1} & = \langle u_0^n, \phi \rangle_{-1} + \int_0^t [\langle \Delta u_s^n, \phi \rangle_{-1} - \langle (u_s^n \cdot \nabla) u_s^n, \phi \rangle_{-1}] ds + \langle L_t^n, \phi \rangle_{-1} \\
& = \langle u_0^n, \phi \rangle_{-1} + \int_0^t [\langle u_s^n, \Delta \phi \rangle_{-1} + \langle u_s^n \otimes u_s^n, \nabla \phi \rangle_{-1}] ds + \langle L_t^n, \phi \rangle_{-1}.
\end{aligned}$$

Thus, for $\varepsilon > 0$ and any stopping time τ bounded by t , we have

$$\begin{aligned}
\langle u_{\tau+\varepsilon}^n - u_\tau^n, \phi \rangle_{-1} & = \int_\tau^{\tau+\varepsilon} [\langle u_s^n, \Delta \phi \rangle_{-1} + \langle u_s^n \otimes u_s^n, \nabla \phi \rangle_{-1}] ds + \langle L_{\tau+\varepsilon}^n - L_\tau^n, \phi \rangle_{-1} \\
& \leq \varepsilon \sup_{s \in [0, t]} (\|u_s^n\|_0 \cdot \|\phi\|_0 + \|u_s^n\|_0^2 \cdot \|\nabla(-\Delta)^{-1} \phi\|_\infty)
\end{aligned}$$

$$+ \sum_{j=1}^n |\beta_j| \cdot |L_{\tau+\varepsilon}^{(j)} - L_{\tau}^{(j)}| \cdot \|(-\Delta)^{-1}\phi\|_0.$$

Using $(a+b)^\theta \leq a^\theta + b^\theta$ provided that $\theta \in (0, 1]$, we get

$$\mathbb{E}|\langle u_{\tau+\varepsilon}^n - u_{\tau}^n, \phi \rangle_{-1}|^{\theta/2} \leq C_\phi \mathbb{E} \left(\sup_{s \in [0, t]} \|u_s^n\|_0^\theta + 1 \right) \varepsilon^{\theta/2} + C_\phi \left(\mathbb{E} \sum_{j=1}^n |\beta_j|^\theta \cdot |L_{\tau+\varepsilon}^{(j)} - L_{\tau}^{(j)}|^\theta \right)^{1/2}$$

By the strong Markov property of Lévy process (cf. [13, p.278, Theorem 40.10]), we have

$$\mathbb{E}|L_{\tau+\varepsilon}^{(j)} - L_{\tau}^{(j)}|^\theta = \mathbb{E}|L_{\varepsilon}^{(j)}|^\theta = \mathbb{E}|L_{\varepsilon}^{(1)}|^\theta, \quad \forall j \in \mathbb{N}.$$

Thus, by (3.6) and (\mathbf{H}_θ) ,

$$\mathbb{E}|\langle u_{\tau+\varepsilon}^n - u_{\tau}^n, \phi \rangle_{-1}|^{\theta/2} \leq C \left[\varepsilon^{\theta/2} + (\mathbb{E}|L_{\varepsilon}^{(1)}|^\theta)^{1/2} \right], \quad (3.7)$$

where the constant C is independent of n, τ and ε . On the other hand, by (2.1), we have

$$\mathbb{E} \left(\sup_{s \in [0, t]} \sum_{j=m}^{\infty} \langle u_s^n, e_j \rangle_{-1}^2 \right)^{\theta/2} = \mathbb{E} \left(\sup_{s \in [0, t]} \sum_{j=m}^{\infty} \frac{\langle u_s^n, e_j \rangle_0^2}{\lambda_j^2} \right)^{\theta/2} \leq \frac{1}{\lambda_m^\theta} \mathbb{E} \left(\sup_{s \in [0, t]} \|u_s^n\|_0^\theta \right). \quad (3.8)$$

By Theorem 2.1 and (3.6)-(3.8), one knows that the law of $(u_t^n)_{t \geq 0}$ in $\mathbb{D}(\mathbb{R}_+; \mathbb{H}^{-1})$ denoted by P_n is tight.

(Step 3): Let P be any accumulation point of $\{P_n, n \in \mathbb{N}\}$. In this step, we show that P is a weak solution of equation (1.1) in the sense of Definition 3.1. First of all, by Skorohod's embedding theorem, there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and $\mathbb{D}(\mathbb{R}_+; \mathbb{H}^{-1})$ -valued random variables X^n and X such that

- (i) Law of X^n under \tilde{P} is P_n and law of X under \tilde{P} is P .
- (ii) X^n converges to X in $\mathbb{D}(\mathbb{R}_+; \mathbb{H}^{-1})$ a.s. as $n \rightarrow \infty$.

Thus, by (3.6), we have

$$\tilde{\mathbb{E}} \left(\sup_{t \in [0, T]} \|X_t^n\|_0^\theta \right) + \tilde{\mathbb{E}} \left(\int_0^T \frac{\theta \|\nabla X_s^n\|_0^2}{(\|X_s^n\|_0^2 + 1)^{1-\theta/2}} ds \right) \leq C(1 + \|\varphi\|_0^\theta + T). \quad (3.9)$$

By Lemma 2.3 and Fatou's lemma, for any $m \in \mathbb{N}$, we have

$$\begin{aligned} \mathbb{E}^P \left(\sup_{t \in [0, T]} \|u_t\|_0^\theta \right) &= \tilde{\mathbb{E}} \left(\sup_{t \in [0, T]} \|X_t\|_0^\theta \right) \leq \lim_{n \rightarrow \infty} \tilde{\mathbb{E}} \left(\sup_{t \in [0, T+1/m]} \|X_t^n\|_0^\theta \right) \\ &\leq (\|\varphi\|_0^2 + 1)^{\theta/2} + C(T + 1/m) + C(T + 1/m)^{1/2}. \end{aligned} \quad (3.10)$$

On the other hand, for any $\delta \in (0, \theta/4)$, by Hölder's inequality and (3.9), we have

$$\begin{aligned} \tilde{\mathbb{E}} \left(\int_0^T \|X_s^n - X_s\|_0^\delta ds \right) &\leq \tilde{\mathbb{E}} \left(\int_0^T \|X_s^n - X_s\|_{-1}^{\delta/2} \|X_s^n - X_s\|_1^{\delta/2} ds \right) \\ &\leq \left(\tilde{\mathbb{E}} \int_0^T \|X_s^n - X_s\|_{-1}^\delta ds \right)^{1/2} \left(\tilde{\mathbb{E}} \int_0^T \|X_s^n - X_s\|_1^\delta ds \right)^{1/2} \rightarrow 0. \end{aligned}$$

So, there exists a subsequence still denoted by n such that for $\tilde{P} \times dt$ -almost all (ω, s) , $X_s^n(\omega)$ converges to $X_s(\omega)$ in \mathbb{H}^0 . By Lemma 2.3 and (3.9), we then obtain

$$\begin{aligned} \mathbb{E}^P \left(\int_0^T \frac{\theta \|\nabla u_s\|_0^2}{(\|u_s\|_0^2 + 1)^{1-\theta/2}} ds \right) &= \tilde{\mathbb{E}} \left(\int_0^T \frac{\theta \|\nabla X_s\|_0^2}{(\|X_s\|_0^2 + 1)^{1-\theta/2}} ds \right) \\ &\leq \lim_{n \rightarrow \infty} \tilde{\mathbb{E}} \left(\int_0^T \frac{\theta \|\nabla X_s^n\|_0^2}{(\|X_s^n\|_0^2 + 1)^{1-\theta/2}} ds \right) \end{aligned}$$

$$\leq C(1 + \|\varphi\|_0^\theta + T). \quad (3.11)$$

Combining (3.10) and (3.11) gives (1.2). In particular, $\sup_{t \in [0, T]} \|u_t\|_0$ and $\int_0^T \frac{\theta \|\nabla u_s\|_0^2}{(\|u_s\|_0^2 + 1)^{1-\theta/2}} ds$ is finite P -almost surely, which produces (i) of Definition 3.1.

Fixing $j \in \mathbb{N}$, in order to show that $M_t^{(j)}$ defined by (3.1) is a Lévy process, we only need to prove that for any $0 \leq s < t$,

$$\mathbb{E}^P e^{i\xi(M_t^{(j)} - M_s^{(j)})} = \tilde{\mathbb{E}} e^{i\xi(\tilde{M}_t^{(j)} - \tilde{M}_s^{(j)})} = \exp \left\{ (t-s) \int_{\mathbb{R} \setminus \{0\}} (e^{i\xi y \beta_j} - 1 - 1_{|y| \leq 1} i\xi y \beta_j) \nu(dy) \right\}, \quad (3.12)$$

where

$$\tilde{M}_t^{(j)} := \langle X_t, e_j \rangle_0 - \langle X_0, e_j \rangle_0 - \int_0^t [\langle X_r, \Delta e_j \rangle_0 + \langle X_r \otimes X_r, \nabla e_j \rangle_0] dr.$$

Fix $0 \leq s < t$ below. By Theorem 2.2, there exists $(s_n, t_n) \downarrow (s, t)$ such that

$$\lim_{n \rightarrow \infty} \tilde{\mathbb{E}} e^{i\xi \langle X_{t_n}^n, e_j \rangle_0} = \tilde{\mathbb{E}} e^{i\xi \langle X_t, e_j \rangle_0}, \quad \lim_{n \rightarrow \infty} \tilde{\mathbb{E}} e^{i\xi \langle X_{s_n}^n, e_j \rangle_0} = \tilde{\mathbb{E}} e^{i\xi \langle X_s, e_j \rangle_0}.$$

By equation (3.2), it is well-known that for any $n \geq j$,

$$\begin{aligned} & \tilde{\mathbb{E}} \exp \left\{ i\xi \left[\langle X_{t_n}^n - X_{s_n}^n, e_j \rangle_0 - \int_{s_n}^{t_n} [\langle X_r^n, \Delta e_j \rangle_0 + \langle X_r^n \otimes X_r^n, \nabla e_j \rangle_0] dr \right] \right\} \\ &= \mathbb{E}^P \exp \left\{ i\xi \left[\langle u_{t_n}^n - u_{s_n}^n, e_j \rangle_0 - \int_{s_n}^{t_n} [\langle u_r^n, \Delta e_j \rangle_0 + \langle u_r^n \otimes u_r^n, \nabla e_j \rangle_0] dr \right] \right\} \\ &= \exp \left\{ (t_n - s_n) \int_{\mathbb{R} \setminus \{0\}} (e^{i\xi y \beta_j} - 1 - 1_{|y| \leq 1} i\xi y \beta_j) \nu(dy) \right\}. \end{aligned}$$

Thus, for proving (3.12), it suffices to prove the following limits:

$$\begin{aligned} & \lim_{n \rightarrow \infty} \tilde{\mathbb{E}} \left| \exp \left\{ i\xi \int_s^t \langle X_r^n \otimes X_r^n, \nabla e_j \rangle_0 dr \right\} - \exp \left\{ i\xi \int_s^t \langle X_r \otimes X_r, \nabla e_j \rangle_0 dr \right\} \right| = 0, \\ & \lim_{n \rightarrow \infty} \tilde{\mathbb{E}} \left| \exp \left\{ i\xi \int_s^t \langle X_r^n, \Delta e_j \rangle_0 dr \right\} - \exp \left\{ i\xi \int_s^t \langle X_r, \Delta e_j \rangle_0 dr \right\} \right| = 0, \\ & \lim_{n \rightarrow \infty} \tilde{\mathbb{E}} \left| \exp \left\{ i\xi \int_{s_n}^{t_n} \langle X_r^n \otimes X_r^n, \nabla e_j \rangle_0 dr \right\} - \exp \left\{ i\xi \int_s^t \langle X_r^n \otimes X_r^n, \nabla e_j \rangle_0 dr \right\} \right| = 0, \\ & \lim_{n \rightarrow \infty} \tilde{\mathbb{E}} \left| \exp \left\{ i\xi \int_{s_n}^{t_n} \langle X_r^n, \Delta e_j \rangle_0 dr \right\} - \exp \left\{ i\xi \int_s^t \langle X_r^n, \Delta e_j \rangle_0 dr \right\} \right| = 0. \end{aligned}$$

Let us only prove the first limit, the others are similar. Noticing that for any $\delta \in (0, 1)$ and $a, b \in \mathbb{R}$,

$$|e^{ia} - e^{ib}| \leq 2(|a - b| \wedge 1) \leq 2|a - b|^\delta,$$

by Hölder's inequality and $\|u\|_0 \leq \|u\|_{-1}^{1/2} \|u\|_1^{1/2}$, we have for $\delta < \theta/4$,

$$\begin{aligned} & \tilde{\mathbb{E}} \left| \exp \left\{ i\xi \int_s^t \langle X_r^n \otimes X_r^n, \nabla e_j \rangle_0 dr \right\} - \exp \left\{ i\xi \int_s^t \langle X_r \otimes X_r, \nabla e_j \rangle_0 dr \right\} \right| \\ & \leq 2|\xi|^\delta \tilde{\mathbb{E}} \left| \int_s^t \langle X_r^n \otimes X_r^n - X_r \otimes X_r, \nabla e_j \rangle_0 dr \right|^\delta \\ & \leq C \tilde{\mathbb{E}} \left(\int_s^t \|X_r^n - X_r\|_0 (\|X_r^n\|_0 + \|X_r\|_0) dr \right)^\delta \\ & \leq C \tilde{\mathbb{E}} \left(\sup_{r \in [s, t]} (\|X_r^n\|_0 + \|X_r\|_0) \int_s^t \|X_r^n - X_r\|_{-1}^{1/2} \|X_r^n - X_r\|_1^{1/2} dr \right)^\delta \end{aligned}$$

$$\begin{aligned}
&\leq C \tilde{\mathbb{E}} \left(\sup_{r \in [s, t]} (\|X_r^n\|_0 + \|X_r\|_0 + 1)^{2\delta - (\theta\delta/2)} \left(\int_s^t \|X_r^n - X_r\|_{-1} dr \right)^{\delta/2} \right. \\
&\quad \times \left. \left(\int_s^t \frac{(\|X_r^n\|_1 + \|X_r\|_1)}{(\|X_r^n\|_0^2 + \|X_r\|_0^2 + 1)^{1-\theta/2}} dr \right)^{\delta/2} \right) \\
&\leq C \left[\tilde{\mathbb{E}} \left(\int_s^t \|X_r^n - X_r\|_{-1} dr \right)^{2\delta} \right]^{1/4} \rightarrow 0,
\end{aligned}$$

as $n \rightarrow \infty$, where in the last inequality, we have used (3.9) and Hölder's inequality. As for the independence of $M^{(j)}$ for different $j \in \mathbb{N}$, it can be proved in a similar way.

Proof of Theorem 1.1: The pathwise uniqueness follows by the classical result for 2D deterministic Navier-Stokes equation. As for the existence of invariant measures, basing on (1.2) (see Remark 1.3), it follows by the classical Bogoliubov-Krylov's argument.

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